Image Processing

- Introduction to 2D signal processing
- 2D Fourier Representations
Signals

Signals are functions that are used to convey information.
– may have 1 or 2 or 3 or even more independent variables

2D signals are much like 1D signals, but there are a few differences.
Signals

Signals are functions that are used to convey information. – **dependent variable** can be a scalar or a vector

**scalar**: brightness at each point \((x, y)\)

**vector**: \((\text{red, green, blue})\) at each point \((x, y)\)

Why do we represent color as a dependent variable?
Signals

Signals are functions that are used to convey information. 
- **dependent variable** can be real, imaginary, or complex-valued.

\[ x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t \]
Signals

Continuous “time” (CT) versus discrete “time” (DT)

Signals from physical systems are often of **continuous** domain:
- continuous time – measured in seconds
- continuous spatial coordinates – measured in meters

Computations usually manipulate functions of **discrete** domain:
- discrete time – measured in samples
- discrete spatial coordinates – measured in samples
Signals

Sampling: converting CT signals to DT

$x(t)$

$x[n] = x(nT)$

$T =$ sampling interval

Important for computational manipulation of physical data.

- digital representations of audio signals (as in MP3)
- digital representations of images (as in JPEG)
Signals

Reconstruction: converting DT signals to CT
zero-order hold

\[ x[n] \]

\[ x(t) \]

\[ T = \text{sampling interval} \]

commonly used in audio output devices
Signals

Reconstruction of 2D DT signals to 2D CT signals. Notion of “nearest neighbor” is more complicated in 2D than in 1D.

Nearest neighbors on a rectangular grid → square pixels.

Nearest neighbors on a non-rectangular grid → “Voronoi” diagram.

Imaging modalities such as MRI give rise to non-rectangular grids.
Signals

Reconstruction: converting DT signals to CT
piecewise linear

\[ x[n] \]

\[ x(t) \]

\[ T = \text{sampling interval} \]
Signals

Reconstruction of 2D DT signals to 2D CT signals. Linear interpolation → “bilinear” interpolation.
Signals

Reconstruction of 2D DT signals to 2D CT signals. Linear interpolation → “bilinear” interpolation.
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Reconstruction of 2D DT signals to 2D CT signals. Linear interpolation $\rightarrow$ “bilinear” interpolation.
Signals

Reconstruction of 2D DT signals to 2D CT signals. Linear interpolation → “bilinear” interpolation.
Reconstruction of 2D DT signals to 2D CT signals. Linear interpolation → “bilinear” interpolation.

The order of interpolation does not affect the result.
Bilinear Interpolation

The order of interpolation does not affect the result.

Bilinear interpolation to find \( f[\alpha, \beta] \) from \( f[0, 0], f[0, 1], f[1, 0], f[1, 1] \).

\[
f[\alpha, \beta] = (1-\alpha) \left( (1-\beta)f[0,0] + \beta f[0,1] \right) + \alpha \left( (1-\beta)f[1,0] + \beta f[1,1] \right)
\]

interpolate along \( x=0 \)

interpolate along \( x=1 \)

interpolate along \( y=(1-\beta) \)

\[
f[\beta, \alpha] = (1-\beta) \left( (1-\alpha)f[0,0] + \alpha f[1,0] \right) + \beta \left( (1-\alpha)f[0,1] + \alpha f[1,1] \right)
\]

interpolate along \( y=0 \)

interpolate along \( y=1 \)

interpolate along \( x=(1-\alpha) \)

\[
= (1-\alpha)(1-\beta)f[0, 0] + (1-\alpha)\beta f[0, 1] + \alpha(1-\beta)f[1, 0] + \alpha\beta f[1, 1]
\]
Signals

Reconstruction of 2D DT signals to 2D CT signals.
Linear interpolation $\rightarrow$ “bilinear” interpolation.

Bilinear interpolation:
- piecewise linear slices along $x$ and $y$ axes
- not generally linear along other axes (not a planar surface)
- the surface is quadratic
Check Yourself

Bilinear interpolation is applied to each of these functions:

\begin{align*}
\begin{array}{c}
\text{\(f_A\)} \\
\begin{array}{ccc}
4 & 32 & 32 \\
0 & 0 & 0 \\
0 & 4 & x
\end{array} \\
\end{array} & \begin{array}{c}
\text{\(f_B\)} \\
\begin{array}{ccc}
4 & 32 & 32 \\
0 & 0 & 32 \\
0 & 4 & x
\end{array} \\
\end{array} & \begin{array}{c}
\text{\(f_C\)} \\
\begin{array}{ccc}
4 & 0 & 32 \\
0 & 0 & 0 \\
0 & 4 & x
\end{array} \\
\end{array} & \begin{array}{c}
\text{\(f_D\)} \\
\begin{array}{ccc}
4 & 16 & 32 \\
0 & 0 & 32 \\
0 & 4 & x
\end{array} \\
\end{array} & \begin{array}{c}
\text{\(f_E\)} \\
\begin{array}{ccc}
4 & 24 & 32 \\
0 & 0 & 24 \\
0 & 4 & x
\end{array} \\
\end{array}
\end{align*}

Which of the functions have the following diagonal values:

\[ f_{i[1,1]} = 11; \quad f_{i[2,2]} = 20; \quad f_{i[3,3]} = 27. \]

1. C
2. E
3. D and E
4. C and D and E
5. none of the above
Bilinear interpolation is applied to each of these functions:

\[ f_A \]
\[ f_B \]
\[ f_C \]
\[ f_D \]
\[ f_E \]

Which of the functions have the following diagonal values:

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1. C
2. E
3. D and E
4. C and D and E
5. none of the above
Fourier Representations

One dimensional and two dimensional CTFT.

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \]

\[ F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j(\omega_xx + \omega_yy)} \, dx \, dy \]
One dimensional and two dimensional DTFT.

\[
F(\Omega) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}
\]

\[
F(\Omega_x, \Omega_y) = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} f[n_x, n_y] e^{-j(\Omega_x n_x + \Omega_y n_y)}
\]
Fourier Representations

One dimensional and two dimensional DFT.

\[
F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi kn}{N}}
\]

\[
F[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f[n_x, n_y] e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}
\]
Orthogonality

Analysis and synthesis equations are similar in 1D and 2D because the basis functions in each version form an orthogonal set.

Two functions are orthogonal if their inner product is zero.

1D DFT basis functions: \( \phi_k[n] = e^{-j \frac{2\pi k}{N} n} \)

Orthogonality of 1D DFT basis functions:

\[
\sum_{n} \phi^*_k[n] \phi_l[n] = \sum_{n=0}^{N-1} e^{j \frac{2\pi k}{N} n} e^{-j \frac{2\pi l}{N} n} = \sum_{n=0}^{N-1} e^{j \frac{2\pi (k-l)}{N} n} = \begin{cases} 
N & \text{if } k = l \\
0 & \text{otherwise}
\end{cases}
\]
Orthogonality

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\]

2D DFT basis functions: \( \phi_{kx,ky}[n_x,n_y] = e^{-j \frac{2\pi k_x}{N_x} n_x} e^{-j \frac{2\pi k_y}{N_y} n_y} \)

Orthogonality of 2D DFT basis functions:
\[
\sum_{n_x, n_y} \phi^*_{kx,ky}[n_x,n_y] \phi_{lx,ly}[n_x,n_y] = \sum_{n_x, n_y} e^{j \frac{2\pi k_x}{N_x} n_x} e^{j \frac{2\pi k_y}{N_y} n_y} e^{-j \frac{2\pi l_x}{N_x} n_x} e^{-j \frac{2\pi l_y}{N_y} n_y} = \sum_{n_x} e^{j \frac{2\pi (k_x-l_x)}{N_x} n_x} \sum_{n_y} e^{j \frac{2\pi (k_y-l_y)}{N_y} n_y} = \begin{cases} N_x N_y & \text{if } k_x = l_x \text{ and } k_y = l_y \\ 0 & \text{otherwise} \end{cases}
\]
The 2D Fourier basis functions have the form

\[ \phi_{k_x,k_y}[n_x,n_y] = e^{-j \frac{2\pi k_x}{N_x} n_x} e^{-j \frac{2\pi k_y}{N_y} n_y} \]

Which (if any) of the following images show the real part of one of the basis functions \( \phi_{k_x,k_y}[n_x,n_y] \)?

A  

B  

C  

D  

What values of \( k_x \) and \( k_y \) correspond to basis function?
Check Yourself

The 2D Fourier basis functions have the form

\[ \phi_{k_x,k_y}[n_x,n_y] = e^{-j\frac{2\pi k_x}{N_x} n_x} e^{-j\frac{2\pi k_y}{N_y} n_y} \]

Which (if any) of the following images show the real part of one of the basis functions \( \phi_{k_x,k_y}[n_x,n_y] \)? A and B

What values of \( k_x \) and \( k_y \) correspond to basis function?

A: (3,4); B: (4,3); C: none; D: none
2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.

\[ f_0[n_x, n_y] = \delta[n_x] \delta[n_y] = \begin{cases} 
1 & n_x = 0 \text{ and } n_y = 0 \\
0 & \text{otherwise} 
\end{cases} \]

\[
F_0[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} \delta[n_x] \delta[n_y] e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)} 
\]

\[
= \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} e^{-j\left(\frac{2\pi k_x}{N_x} 0 + \frac{2\pi k_y}{N_y} 0\right)} 
\]

\[
= \frac{1}{N_x N_y} 
\]

\[
\delta[n_x] \delta[n_y] \overset{\text{DFT}}{\Rightarrow} \frac{1}{N_x N_y} 
\]
2D Discrete Fourier Transform

Alternatively, implement a 2D DFT as a sequence of 1D DFTs.

\[
F[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f[n_x, n_y] e^{-j \left( \frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y \right)}
\]

\[
= \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} \left( \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} f[n_x, n_y] e^{-j \frac{2\pi k_x}{N_x} n_x} \right) e^{-j \frac{2\pi k_y}{N_y} n_y}
\]

first take DFTs of rows

then take DFTs of resulting columns

Could just as well start with columns and then do rows.
Example: Find the DFT of a 2D unit sample.

\[ f[n_x, n_y] \]
Example: Find the DFT of a 2D unit sample.
Example: Find the DFT of a 2D unit sample.
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Example: Find the DFT of a 2D unit sample.
2D Discrete Fourier Transform

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2D Discrete Fourier Transform

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2D Discrete Fourier Transform

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2D Discrete Fourier Transform

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2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.
2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.

\[ f[n_x, n_y] \]

Magnitude

\[ \text{DFT(rows)} \]

Angle

\[ F[k_x, k_y] \]
2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.

\[ f[n_x, n_y] \]

\[ n \quad \text{DFT(rows)} \]

\[ k_y \quad F[k_x, k_y] \]
Example: Find the DFT of a 2D unit sample.
Example: Find the DFT of a 2D unit sample.

Magnitude

Angle

$F[k_x, k_y]$
2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.
Example: Find the DFT of a 2D unit sample.
2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.

\[ f[n_x, n_y] \quad \xrightarrow{\text{DFT}} \quad F[k_x, k_y] \]
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[ f_{v}[n_x, n_y] = \delta[n_x] = \begin{cases} 
1 & n_x = 0 \\
0 & \text{otherwise}
\end{cases} \]

\[ F_{v}[k_x, k_y] = \frac{1}{N_{x}N_{y}} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} \delta[n_x] e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)} \]

\[ = \frac{1}{N_{x}N_{y}} \sum_{n_x=0}^{0} \sum_{n_y=0}^{N_y-1} e^{-j\left(\frac{2\pi k_x}{N_x} 0 + \frac{2\pi k_y}{N_y} n_y\right)} = \frac{1}{N_{x}N_{y}} \sum_{n_y=0}^{N_y-1} e^{-j\frac{2\pi k_y}{N_y} n_y} \]

But \[ \sum_{n_y=0}^{N_y-1} e^{-j\frac{2\pi k_y}{N_y} n_y} = \begin{cases} 
N_y & k_y = 0 \\
0 & \text{otherwise}
\end{cases} \]

\[ F_{v}[k_x, k_y] = \frac{1}{N_{x}N_{y}} N_y \delta[k_y] = \frac{1}{N_x} \delta[k_y] \]

\[ \delta[n_x] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{N_x} \delta[k_y] \]
Example: Find the DFT of a vertical line.

\[ f[n_x, n_y] \]
Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.

**2D Discrete Fourier Transform**

- **Magnitude**
- **Angle**
Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[ f[n_x, n_y] \]

Magnitude

Angle

DFT(rows)
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[ f[n_x, n_y] \quad \text{DFT(rows)} \]

Magnitude

Angle

\[ f[n_x, n_y] \]

\[ n \]

\[ n_x \]

\[ k_x \]

\[ n_y \]

\[ n \]

\[ n_x \]

\[ k_x \]
Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[
\begin{align*}
&n_y \quad f[n_x, n_y] \\
&\text{Magnitude} \\
\end{align*}
\]

\[
\begin{align*}
&n \\
&DFT(\text{rows}) \\
&n_x \quad k_x \\
\end{align*}
\]

\[
\begin{align*}
&n_y \\
&\text{Angle} \\
\end{align*}
\]

\[
\begin{align*}
&n \\
\end{align*}
\]

\[
\begin{align*}
&n_x \quad k_x \\
\end{align*}
\]
Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.
Example: Find the DFT of a vertical line.

\[ f[n_x, n_y] \rightarrow n_x \rightarrow \text{DFT(rows)} \rightarrow k_x \rightarrow F[k_x, k_y] \]
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

- $f[n_x, n_y]$ in the spatial domain
- $F[k_x, k_y]$ in the frequency domain

Magnitude and Angle representations:
- Spatial Domain
- Frequency Domain

DFT (rows)

$n_x$ $k_x$

$F[k_x, k_y]$
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[ f[n_x, n_y] \]
Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.
2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

\[
\begin{align*}
\text{Magnitude} & \\
n_y & f[n_x, n_y] & \overset{\text{DFT}}{\leftrightarrow} & k_y & F[k_x, k_y] \\
\text{Angle} & \\
n_y & & & k_y & F[k_x, k_y]
\end{align*}
\]
Example: Find the DFT of a horizontal line.

\[ f_h[n_x,n_y] = \delta[n_y] = \begin{cases} 
1 & n_y = 0 \\
0 & \text{otherwise} 
\end{cases} \]

\[ F_h[k_x,k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} \delta[n_y] e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)} \]

\[ = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{0} e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} 0\right)} = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} e^{-j\frac{2\pi k_x}{N_x} n_x} \]

But \[ \sum_{n_x=0}^{N_x-1} e^{-j\frac{2\pi k_x}{N_x} n_x} = \begin{cases} 
N_x & k_x = 0 \\
0 & \text{otherwise} 
\end{cases} \]

\[ F_h[k_x,k_y] = \frac{1}{N_x N_y} N_x \delta[k_x] = \frac{1}{N_y} \delta[k_x] \]

\[ \delta[n_y] \overset{\text{DFT}}{\iff} \frac{1}{N_y} \delta[k_x] \]
Example: Find the DFT of a horizontal line.

\[
f[n_x, n_y]
\]
Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.

2D Discrete Fourier Transform

- **Magnitude**
  - $f[n_x, n_y]$ (input)
  - DFT(rows) (output)

- **Angle**
  - $f[n_x, n_y]$ (input)
  - DFT(rows) (output)
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.

\[ f[n_x, n_y] \]

\[ \text{DFT(rows)} \]
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.

- **Magnitude**: Shows the magnitude of the DFT for different frequencies. The horizontal line corresponds to a peak in the magnitude spectrum at a specific frequency.
- **Angle**: Displays the phase angle of the DFT. The horizontal line indicates a phase shift in the frequency domain.
Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.

\[ f[n_x, n_y] \]

\[ DFT(\text{rows}) \]

\[ F[k_x, k_y] \]
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.

\[ f[n_x, n_y] \]

Magnitude

\[ f[n_x, n_y] \]

Angle

\[ F[k_x, k_y] \]
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.

\[ f[n_x, n_y] \]

Magnitude

\[ \text{DFT(rows)} \]

Angle

\[ F[k_x, k_y] \]
Example: Find the DFT of a horizontal line.
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.
Example: Find the DFT of a horizontal line.

$$f[n_x, n_y]$$

DFT(rows)

$$F[k_x, k_y]$$
2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.

```
n_y \begin{array}{cc} f[n_x, n_y] \\ \end{array} \xrightarrow{\text{DFT}} \begin{array}{cc} k_y \\ F[k_x, k_y] \end{array}
```

- **Magnitude**
  - Input: $f[n_x, n_y]$
  - Output: $F[k_x, k_y]$

- **Angle**
  - Input: $f[n_x, n_y]$
  - Output: $F[k_x, k_y]$
2D Discrete Fourier Transform

Example: Find the DFT of a constant.

\[ f_1[n_x, n_y] = 1 \]

\[ F_1[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} e^{-j \left( \frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y \right)} \]

\[ = \left( \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} e^{-j \frac{2\pi k_x}{N_x} n_x} \right) \left( \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} e^{-j \frac{2\pi k_y}{N_y} n_y} \right) \]

\[ = \delta[k_x] \delta[k_y] \]

\[ 1 \overset{\text{DFT}}{\leftrightarrow} \delta[k_x] \delta[k_y] \]
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.

\[ f[n_x, n_y] \]

Magnitude

Angle

DFT(rows)

\[ F[k_x, n_y] \]

\[ F[n_x, k_x] \]
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.

\[ f[n_x, n_y] \]

\[ F[k_x, k_y] \]

Magnitude

Angle

DFT(rows)
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.

Magnitude

Angle

F[nx, ny]

DFT(rows)

F[kx, ky]
Example: Find the DFT of a constant.
Example: Find the DFT of a constant.
2D Discrete Fourier Transform

Example: Find the DFT of a constant.
Translating (Shifting) an Image

Effect of image translation (shifting) on its Fourier transform.

Assume that $f_0[n_x, n_y] \xrightarrow{\text{DFT}} F_0[k_x, k_y]$.

Find the 2D DFT of $f_1[n_x, n_y] = f_0[n_x - n_{x0}, n_y - n_{y0}]$

$$F_1[k_x, k_y] = \sum_{k_x} \sum_{k_y} f_1[n_x, n_y] e^{-j \frac{2\pi k_x}{N_x} n_x} e^{-j \frac{2\pi k_y}{N_y} n_y}$$

$$= \sum_{k_x} \sum_{k_y} f_0[n_x - n_{x0}, n_y - n_{y0}] e^{-j \frac{2\pi k_x}{N_x} n_x} e^{-j \frac{2\pi k_y}{N_y} n_y}$$

Let $l_x = n_x - n_{x0}$ and $l_y = n_y - n_{y0}$. Then

$$F_1[k_x, k_y] = \sum_{l_x} \sum_{l_y} f_0[l_x, l_y] e^{-j \frac{2\pi k_x}{N_x} (l_x + n_{x0})} e^{-j \frac{2\pi k_y}{N_y} (l_y + n_{y0})}$$

$$= e^{-j \frac{2\pi k_x}{N_x} n_{x0}} e^{-j \frac{2\pi k_y}{N_y} n_{y0}} F_0[k_x, k_y]$$

Translating an image adds linear (in $k_x, k_y$) phase to its transform.
2D Discrete Fourier Transform

Example: Find the DFT of a shifted 2D unit sample.
Example: Find the DFT of a shifted 2D unit sample.

\[
\begin{align*}
n_y & \quad f[n_x, n_y] \\
& \quad \text{DFT} \\
& \quad k_y & \quad F[k_x, k_y]
\end{align*}
\]
Example: Find the DFT of a shifted 2D unit sample.

\[ f[n_x, n_y] \xrightarrow{\text{DFT}} F[k_x, k_y] \]
Example: Find the DFT of a shifted 2D unit sample.
2D Discrete Fourier Transform

Example: Find the DFT of a shifted 2D unit sample.

\[ f[n_x, n_y] \xrightarrow{\text{DFT}} F[k_x, k_y] \]
Example: Find the DFT of a shifted 2D unit sample.

\[ f[n_x, n_y] \xrightarrow{\text{DFT}} F[k_x, k_y] \]
Example: Find the DFT of a shifted 2D unit sample.
Example: Find the DFT of a shifted 2D unit sample.
2D Discrete Fourier Transform

Example: Find the DFT of a shifted 2D unit sample.
Example: Find the DFT of a shifted 2D unit sample.

\[
\begin{align*}
\text{Magnitude:} & & \text{Angle:} \\
\begin{bmatrix} f[n_x, n_y] \end{bmatrix} & \overset{\text{DFT}}{\leftrightarrow} & \begin{bmatrix} F[k_x, k_y] \end{bmatrix} \\
\end{align*}
\]
The signal $f_s[n_x, n_y]$ is defined for $-5 \leq n_x \leq 5$ and $-5 \leq n_y \leq 5$ and is 1 at a single point $[n_x = 1, n_y = 1]$.

How many of the following are true for $F_s[k_x, k_y]$?

1. $\angle F_s[k, 0] = -\angle F_s[0, -k] \quad \forall k$
2. $\angle F_s[5, 5] = \angle F_s[5, 0] + \angle F_s[0, 5]$
3. $\angle F_s[0, 0] = 0$
The signal \( f_s[n_x, n_y] \) is defined for \(-5 \leq n_x \leq 5\) and \(-5 \leq n_y \leq 5\) and is 1 at a single point \([n_x=1, n_y=1]\).

How many of the following are true for \( F_s[k_x, k_y] \)? 3

1. \( \angle F_s[k, 0] = -\angle F_s[0, -k] \) \( \forall k \)
2. \( \angle F_s[5, 5] = \angle F_s[5, 0] + \angle F_s[0, 5] \)
3. \( \angle F_s[0, 0] = 0 \)
Today: Image Processing

- Introduction to 2D signal processing
- 2D Fourier Representations

In lab, we will practice these ideas and apply them to real images.