6.003: Signal Processing

Properties of Fourier Series

- linearity
- time shift
- stretching in time
- even/odd decomposition
- conjugation
- Parseval’s relation

and how they help us understand what we saw (and heard) in lab.

February 20, 2018
Advisory Group

Weekly meetings with class representatives

- help staff understand student perspective
- learn about teaching

Meet on Wednesday afternoon

Interested? ... Send email to freeman@mit.edu
Fourier Series

Signals are functions that convey information.

Continuous-Time Fourier Series

\[ a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} \, dt \]

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

Discrete-Time Fourier Series

\[ a_k = a_k + N = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\Omega_0 n} \]

\[ x[n] = x[n + N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \]

where \( \omega_0 = \frac{2\pi}{T} \)

where \( \Omega_0 = \frac{2\pi}{N} \)

Fourier series provide a different view of that information.
A narrow pulse contains all possible frequencies (design lab 2).

\[ x[n] = x[n + 16] = \delta[n] \]

\[ X[k] = 1/16 \]

What is special about \( n = 0 \) so that \( x[0] \) is big?

What is different about \( n \neq 0 \) so that \( x[n \neq 0] = 0 \)?

If a signal contains all possible frequencies, is it necessarily a pulse?
A Narrow Pulse Contains All Possible Frequencies

Each time sample results from a sum across frequencies ($k$).

$$x[n] = x[n + 16] = \delta[n]$$

$$X[k] = \frac{1}{16}$$

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N} kn}$$

- contributions across $k$ can sum constructively or destructively.
A Narrow Pulse Contains All Possible Frequencies

Alternatively, the function $x[n]$ is the sum of $k$ time functions.

$$x[n] = x[n + 16] = \delta[n]$$

$$X[k] = \frac{1}{16}$$

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi}{N} kn} = X[0] + X[1] e^{j \frac{2\pi}{N} n} + X[2] e^{j \frac{2\pi}{N} 2n} + X[3] e^{j \frac{2\pi}{N} 3n} + \cdots$$

$1 = x[0]$ is special because the peaks of the cosines line up at $n = 0$. 

$$e^{j \frac{2\pi}{N} n} = e^{j \frac{2\pi}{N} 2n} = \cdots$$
Time Delay

How do the Fourier coefficients change if $x[n]$ is delayed in time?

\[
\begin{align*}
x[n] &= x[n + N] \equiv \delta[n] \quad \leftrightarrow \quad a_k \\
x[n] &= x[n + N] \equiv \delta[n - 1] \quad \leftrightarrow \quad b_k
\end{align*}
\]

Find an expression for $b_k$ in terms of $a_k$. 
How do the Fourier coefficients change if \( x[n] \) is delayed in time?

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N} kn}
\]

\[
b_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n-1] e^{-j\frac{2\pi}{N} kn}
\]

Find an expression for \( b_k \) in terms of \( a_k \).

Let \( m = n - 1 \) then

\[
b_k = \frac{1}{N} \sum_{m+1=\langle N \rangle} x[m] e^{-j\frac{2\pi}{N} k(m+1)} = \frac{1}{N} \sum_{m=\langle N \rangle} x[m] e^{-j\frac{2\pi}{N} km} e^{-j\frac{2\pi}{N} k} = e^{-j\frac{2\pi}{N} k} a_k
\]

**Time delay** changes the **phase** of the Fourier components.

The magnitudes of the coefficients are unchanged.
Delaying by one time step adds a phase delay proportional to $k$. 

- Phase delays at each frequency shift that harmonic by 1 time step.
Signals With All Possible Frequencies

Randomizing phase scrambles the time response.

Here the magnitudes of $|X[k]|$ were equal to each other, but the angles of $X[k]$ where chosen randomly.
Sum of Pulses

Adding a train of pulses to a delayed version changes the percept, sometimes drastically (concept lab 1).

\[
\delta[n] + \delta[n - 2]
\]

\[
\delta[n] \quad \xleftrightarrow{FS} \quad \frac{1}{N}
\]

\[
\delta[n - 2] \quad \xleftrightarrow{FS} \quad \frac{1}{N} e^{-j \frac{2\pi}{N} 2k}
\]

\[
\delta[n] + \delta[n - 2] \quad \xleftrightarrow{FS} \quad \frac{1}{N} \left( 1 + e^{-j \frac{2\pi}{N} 2k} \right)
\]  
(linearity of FS)
The Fourier series of a sum of scaled signals is the sum of scaled versions of their Fourier series.

\[ x[n] \overset{\text{FS}}{=} X[k] \]
\[ y[n] \overset{\text{FS}}{=} Y[k] \]
\[ ax[n] + by[n] \overset{\text{FS}}{=} aX[k] + bY[k] \]

Easy to prove from definitions:

\[ X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi}{N} kn} \]
\[ Y[k] = \frac{1}{N} \sum_{n=\langle N \rangle} y[n] e^{-j \frac{2\pi}{N} kn} \]
\[ aX[k] + bY[k] = \frac{1}{N} \sum_{n=\langle N \rangle} (ax[n] + by[n]) e^{-j \frac{2\pi}{N} kn} \]
Adding a train of pulses to a delayed version changes the percept, sometimes drastically (concept lab 1).

\[ \delta[n] + \delta[n-2] \]

\[ \delta[n] \xleftrightarrow{FS} \frac{1}{N} \]
\[ \delta[n-2] \xleftrightarrow{FS} \frac{1}{N} e^{-j \frac{2\pi}{N} 2k} \]
\[ \delta[n] + \delta[n-2] \xleftrightarrow{FS} \frac{1}{N} \left(1 + e^{-j \frac{2\pi}{N} 2k}\right) \quad \text{(linearity of FS)} \]

Fourier components of the sum no longer have equal magnitudes. The component at \( k = 4 \) has zero amplitude.
Adding a train of pulses to a delayed version changes the percept, sometimes drastically (concept lab 1).

Visualize the complex sum on the complex plane.

\[ 1 + e^{-j\frac{2\pi}{16}} \frac{2k}{2k} \]
Sum of Pulses

How do the Fourier coefficients depend on delay?

\[
\delta[n] + \delta[n-2] \quad \overset{\text{FS}}{\Rightarrow} \quad |a_k| \\
\delta[n] + \delta[n-4] \quad \overset{\text{FS}}{\Rightarrow} \\
\delta[n] + \delta[n-6] \quad \overset{\text{FS}}{\Rightarrow} \\
\delta[n] + \delta[n-8] \quad \overset{\text{FS}}{\Rightarrow}
\]

Each delay generates a different distribution of harmonic amplitudes.

When the delay is half a cycle, there are no odd harmonics.

Removing odd harmonics \(\rightarrow\) new point of constructive addition.
Sum of Pulses

Why is there such an abrupt percept change when the delay is $N/2$?

Find $|a_k|$ as a function of $n_0$.

\[ \delta[n] + \delta[n - n_0] \iff a_k = \frac{1}{N} \left(1 + e^{-j \frac{2\pi k n_0}{N}}\right) = \frac{1}{N} \left(1 + \cos \frac{2\pi k n_0}{N} + j \sin \frac{2\pi k n_0}{N}\right) \]

\[ |a_k| = \frac{1}{N} \sqrt{2 + 2 \cos \frac{2\pi k n_0}{N}} \]

Magnitude of fundamental gradually declines to zero at $n_0 = N/2$.

But humans are VERY sensitive to even low level tones!
Properties of Discrete-Time Fourier Series

Relation of coefficients for complex exponential and sinusoidal series.

\[ x[n] = \sum_{k=\langle \! \langle N \rangle \rangle} a_k e^{j \frac{2\pi}{N} nk} = \sum_{k=} c_k \cos \frac{2\pi nk}{N} + \sum_{k=} d_k \sin \frac{2\pi nk}{N} \]

If \( x[n] \) is real-valued, then \( c_k \) and \( d_k \) are real-valued, but \( a_k \) is not!

By orthogonality, we must match components with same frequency:

\[
a_k e^{j \frac{2\pi}{N} nk} + a_{-k} e^{-j \frac{2\pi}{N} nk} = c_k \cos \frac{2\pi kn}{N} + d_k \sin \frac{2\pi kn}{N}
\]

\[
= \frac{c_k}{2} \left( e^{j \frac{2\pi}{N} kn} + e^{-j \frac{2\pi}{N} kn} \right) + \frac{d_k}{2j} \left( e^{j \frac{2\pi}{N} kn} - e^{-j \frac{2\pi}{N} kn} \right)
\]

\[
a_k = \frac{c_k}{2} + \frac{d_k}{2j} = \frac{1}{2}(c_k - jd_k)
\]

\[
a_{-k} = \frac{c_k}{2} - \frac{d_k}{2j} = \frac{1}{2}(c_k + jd_k) = a_k^* \]

Notice that \( a_{-k} = a_k^* \) for real-valued \( x[n] \). (conjugate symmetry)
Properties of Discrete-Time Fourier Series

Drastic changes in phase.

Example: play a song backwards (Donovan).

\[ x[n] \Leftrightarrow_{FS} a_k \]
\[ x[-n] \Leftrightarrow_{FS} b_k \]

Find \( b_k \) in terms of \( a_k \).

\[
  b_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[-n] e^{-j\frac{2\pi}{N} nk}
\]

Let \( m = -n \).

\[
  b_k = \frac{1}{N} \sum_{-m=\langle N \rangle} x[m] e^{j\frac{2\pi}{N} mk} = \frac{1}{N} \sum_{m=\langle N \rangle} x[m] e^{-j\frac{2\pi}{N} m(-k)} = a_{-k}
\]

If \( x[n] \) is real-valued, \( a_{-k} = a_k^* \).

Then \( b_k = a_k^* \rightarrow |b_k| = |a_k| \) and \( \angle b_k = -\angle a_k \).

Playing a sound backwards negates the phase of Fourier series coef’s.
In design lab 2, we "stretched" recorded signals from musical instruments to change their pitch. One stretching method was zero-order hold.

We found that this method led to a number of distortions.

Can we use Fourier series to gain some insight into these distortions?
Properties of Discrete-Time Fourier Series

Compare a sinusoid with a zero-order-hold version of itself.

Remove odd '#d samples

stretch by a factor of 2

and add a delayed copy
Properties of Discrete-Time Fourier Series

Compare a sinusoid with a zero-order-hold version of itself.

\[ x[n] = \cos \frac{2\pi n}{16} \]

Start with the original signal

\[ N = 16 \]

\[ x[n] = \cos \frac{2\pi n}{16} \]

Remove odd #'d samples

\[ N = 8 \]

\[ \cos \frac{2\pi n}{8} \]

Stretch by a factor of 2

\[ N = 16 \]

\[ y[n] \]

\[ y[n] + y[n - 1] \]

and add a delayed copy

\[ |(1 + e^{-j \frac{2\pi}{16} k}) Y[k]| \]
Properties of Discrete-Time Fourier Series

Parseval’s relation.

\[ a_k = \frac{1}{N} \sum_n x[n] e^{-j \frac{2\pi}{N} nk} \]

\[
\sum_k |a_k|^2 = \sum_k a_k a_k^* = \sum_k \frac{1}{N} \left( \sum_m x[m] e^{-j \frac{2\pi}{N} mk} \right) \frac{1}{N} \left( \sum_n x^*[n] e^{-j \frac{2\pi}{N} nk} \right)
\]

\[
= \frac{1}{N^2} \sum_k \sum_m \sum_n x[m] x^*[n] e^{j \frac{2\pi}{N} (n-m)k}
\]

\[
= \frac{1}{N^2} \sum_m \sum_n x[m] x^*[n] \sum_k e^{j \frac{2\pi}{N} (n-m)k}
\]

\[
= \frac{1}{N^2} \sum_m \sum_n x[m] x^*[n] \sum_k N \delta[n - m]
\]

\[
= \frac{1}{N} \sum_n x[n] x^*[n] = \frac{1}{N} \sum_n |x[n]|^2
\]
Even and Odd Decomposition

Breaking a signal into even and odd parts.

Let

\[ x[n] = x_e[n] + x_o[n] \]

where \( x_e[n] \) is an even function of \( n \):

\[ x_e[n] = x_e[-n] \]

and \( x_o[n] \) is an odd function of \( n \):

\[ x_o[n] = -x_o[-n] \]

Find \( x_e[n] \) and \( x_o[n] \).
### Even and Odd Decomposition

Breaking a signal into even and odd parts.

\[
x[n] = x_e[n] + x_o[n]
\]

\[
x[-n] = x_e[-n] + x_o[-n] = x_e[n] - x_o[n]
\]

\[
x[n] + x[-n] = 2x_e[n]
\]

\[
x[n] - x[-n] = 2x_o[n]
\]

\[
x_e[n] = \frac{1}{2}(x[n] + x[-n])
\]

\[
x_o[n] = \frac{1}{2}(x[n] - x[-n])
\]

Is this decomposition **always possible**?

Is this decomposition **unique**?
Even and Odd Decomposition

Find the Fourier series for the even and odd parts of a signal.

\[
x[n] \quad \overset{\text{FS}}{\leftrightarrow} \quad a_k
\]

\[
x[-n] \quad \overset{\text{FS}}{\leftrightarrow} \quad a_{-k}
\]

\[
\text{Ev}(x[n]) = \frac{1}{2}(x[n] + x[-n]) \quad \overset{\text{FS}}{\leftrightarrow} \quad \frac{1}{2}(a_k + a_{-k})
\]

If \(x[n]\) is real, then \(a_{-k} = a_k^\ast\).

\[
\text{Ev}(x[n]) = \frac{1}{2}(x[n] + x[-n]) \quad \overset{\text{FS}}{\leftrightarrow} \quad \frac{1}{2}(a_k + a_k^\ast) = \text{Re}(a_k)
\]

→ The Fourier series for the even part of \(x[n]\) is the real part of the Fourier series for \(x[n]\).

Corollary: real and even \(\overset{\text{FS}}{\leftrightarrow}\) real and even
Check Yourself

Complex numbers.

How many of the following are true?

A. \( \frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta \)

B. \((\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta\)

C. \(|2 + j2 + e^{j\pi/4}| = |2 + j2| + |e^{j\pi/4}|\)

D. \(\text{Im}(j^j) > \text{Re}(j^j)\)

E. \(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} 1\)

0. 0 1. 1 2. 2 3. 3 4. 4 5. 5
Check Yourself

Complex numbers.

How many of the following are true? 4

A. \( \frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta \) \quad \checkmark

B. \((\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta \) \quad \checkmark

C. \(|2 + j2 + e^{j\pi/4}| = |2 + j2| + |e^{j\pi/4}| \) \quad \checkmark

D. \( \text{Im}(j^j) > \text{Re}(j^j) \) \quad \times

E. \( \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} 1 \) \quad \checkmark

0. 0 1. 1 2. 2 3. 3 4. 4 5. 5
What is the fundamental (shortest) period of each of the following signals?

1. \( x_1[n] = \cos \frac{\pi n}{12} \)

2. \( x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} \)

3. \( x_3[n] = \cos n + \cos 2n + \cos 3n \)
Check Yourself

\[ x_1[n] = \cos \frac{\pi n}{12} = \cos \left( \frac{\pi n}{12} + 2\pi \right) = \cos \frac{\pi (n + 24)}{12} = x_1[n + 24] \]

\[ x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} = \cos \left( \frac{\pi n}{12} + 10\pi \right) + 3 \cos \left( \frac{\pi n}{15} + 8\pi \right) \]

\[ = \cos \frac{\pi (n + 120)}{12} + 3 \cos \frac{\pi (n + 120)}{15} = x_2[n + 120] \]

\[ x_3[n] = \cos n + \cos 2n + \cos 3n \text{ is not periodic.} \]
What is the fundamental (shortest) period of each of the following signals?

1. \( x_1[n] = \cos \frac{\pi n}{12} \) \quad 24

2. \( x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} \) \quad 120

3. \( x_3[n] = \cos n + \cos 2n + \cos 3n \) \quad \infty